FINITE SETS

DEFINITION: A set X is said to be **finite** iff $X = \emptyset$ or there exists a bijection $F : X \rightarrowtail\!\!\!\!\twoheadrightarrow \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$.

A set X is called **infinite** iff X is not finite.

FACTS ABOUT FINITE SETS:

- Subsets of finite sets are finite, hence intersections of finite sets are finite.

- Finite unions of finite sets are finite.

- If there exists $F : X \rightarrowtail\!\!\!\!\twoheadrightarrow \{1, 2, \ldots, n\}$ and $G : X \rightarrowtail\!\!\!\!\twoheadrightarrow \{1, 2, \ldots, m\}$, then $n = m$.

EXAMPLE: Let X be a set and define $\mathcal{T}_f = \{T \subseteq X \,:\, T = \emptyset \text{ or } X \backslash T \text{ is finite.}\}$.

- Show $\mathcal{T}_f$ is a topology on X.

  NOTE: $\mathcal{T}_f$ is called the **cofinite** topology.

- If X is finite, what is $\mathcal{T}_f$?

- Under what conditions is a space $(X, \mathcal{T}_f)$ connected?

- What separation properties are satisfied by $(X, \mathcal{T}_f)$ if X is infinite?

COMPACTNESS

RECALL: A family of sets $\mathcal{V}$ is said to **cover** a set X iff $X = \bigcup_{V \in \mathcal{V}} V$. Equivalently:

$$\mathcal{V} \text{ covers } X \iff \forall x \in X,\ \exists\, V \in \mathcal{V} \text{ such that } x \in V.$$

If $(X, \mathcal{T})$ is a topological space, $\mathcal{V}$ is called an **open cover** of X if $\mathcal{V}$ is a cover of open sets. $(\mathcal{V} \subseteq \mathcal{T})$

EXAMPLES: Suppose $(X, \mathcal{T})$ is a topological space. The following are examples of open covers of X:

- $\mathcal{T}$
- $\mathcal{B}$ where $\mathcal{B}$ is any base for $\mathcal{T}$
- if in addition (X, d) is a metric space, then $\mathcal{N} = \{N_1(x, d) \mid x \in X\}$ is an open cover

DEFINITION: A space $(X, \mathcal{T})$ is said to be **compact** iff every open cover $\mathcal{V}$ admits a finite subcover.

That is if $\mathcal{V}$ is an open cover of X, there exist $V_1,\ V_2,\ \ldots V_n \in \mathcal{V}$ such that $X = \bigcup_{i=1}^{n} V_i$.
A 'compact' space may be thought of as being 'topologically small.'

EXAMPLE:

- If X is a finite set, then $(X, \mathcal{T})$ is compact.
- Any set X with the indiscrete topology, $(X, \mathcal{I})$, is compact.
- Any set X with the cofinite topology, $(X, \mathcal{T}_f)$ is compact.
- $\mathbb{R}$ with the usual topology is **not** compact.
- $[0, 1)$ with the usual topology is **not** compact.

EXAMPLE: Compactness is 'closed' hereditary. That is:

$$\text{If } (X, \mathcal{T}) \text{ is compact and } A \subseteq X \text{ is closed, then } (A, \mathcal{T}_A) \text{ is compact.}$$

EXAMPLE: Continuous images of compact spaces are compact. That is:

$$\text{If } (X, \mathcal{T}) \text{ is compact and } F : (X, \mathcal{T}) \twoheadrightarrow (Y, \mathcal{U}) \text{ is continuous then } (Y, \mathcal{U}) \text{ is compact.}$$

EXAMPLE: Compact subsets of Hausdorff spaces are closed. That is:

$$\text{If } (X, \mathcal{T}) \text{ is Hausdorff and } A \subseteq X \text{ is compact, then } A \text{ is closed.}$$

EXAMPLE: If $(X, \mathcal{T})$ is a compact Hausdorff space, then:

- $A \subseteq X$ is compact iff A is closed.

- any strictly coarser topology is compact but not Hausdorff.

- any strictly finer topology is Hausdorff but not compact.

EXAMPLE: If $(X, \mathcal{T})$ is compact and $(Y, \mathcal{U})$ is Hausdorff, then continuous maps from X to Y are closed.

EXAMPLE: Suppose $(X, \mathcal{T})$ is Hausdorff and $A \subseteq X$ is compact.

- If $x_0 \notin A$, there exists open sets T and U with $x_0 \in T$ and $A \subseteq U$ with $T \cap U = \emptyset$.

- If B is compact and $A \cap B = \emptyset$, there exists open sets T and U with $B \subseteq T$ and $A \subseteq U$ with $T \cap U = \emptyset$.

EXAMPLE: Products of compact spaces is compact. That is:

If $(X, \mathcal{T})$ and $(X, \mathcal{T})$ are compact, then is $(X \times Y, \mathcal{T} \times \mathcal{U})$ is compact.

NOTE: This is the (finite) version of the Tychonoff Theorem which is equivalent to the Axiom of Choice!

DEFINITION: A collection of sets $\mathcal{C}$ is said to satisfy the **finite intersection property** or **f.i.p.** if the intersection of any **finite** number of sets in $\mathcal{C}$ is nonempty. That is, if $C_1, C_2, \ldots, C_n \in \mathcal{C}$, then $\displaystyle\bigcap_{i=1}^{n} C_i \neq \emptyset$.

THEOREM: $(X, \mathcal{T})$ is compact iff $\mathcal{C}$ is a collection of closed sets with f.i.p. $\implies \displaystyle\bigcap_{C \in \mathcal{C}} C \neq \emptyset$

NESTED INTERVAL THEOREM: Suppose $[a_0, b_0] \supseteq [a_1, b_1] \supseteq [a_2, b_2] \ldots$ with $\displaystyle\lim_{i \to \infty} (b_i - a_i) = 0$.

Then there is a real number x such that $\displaystyle\bigcap_{i=1}^{\infty} [a_i, b_i] = \{x\}$.

DEFINITION: A subset $A \subseteq \mathbb{R}$ is said to be **bounded** iff there is a real number M so that $A \subseteq [-M, M]$.

EXAMPLE: Let $\mathbb{R}$ have the Euclidean topology. Prove if $A \subseteq \mathbb{R}$ is compact, then A is closed and bounded.

THEOREM: (Heine-Borel): Let $\mathbb{R}$ have the Euclidean topology. $A \subseteq \mathbb{R}$ is compact iff A is closed and bounded.

NOTE: If we can show the result is true for intervals of the form $[-M, M]$ we are done (why?)

The classic proof uses the Nested Interval Property.

NOTE: If we can show the result is true for $[0, 1]$ we are done (why?)

COROLLARY: The Extreme Value Theorem.

EXAMPLE: The Cantor Set ...